

# Some Siegel threefolds with a Calabi-Yau model II

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## Introduction

In the paper [FSM] we described some Siegel modular threefolds which admit a Calabi-Yau model. Using a different method we give in this paper an enlarged list of such varieties. Basic for our method is the paper [GN] of van Geemen and Nygaard. In this paper they prove that the complete intersection  $\mathcal{X}$  in  $P^7(\mathbb{C})$ , defined by the equations

$$\begin{aligned} \mathcal{X}: \quad Y_0^2 &= X_0^2 + X_1^2 + X_2^2 + X_3^2, \\ Y_1^2 &= X_0^2 - X_1^2 + X_2^2 - X_3^2, \\ Y_2^2 &= X_0^2 + X_1^2 - X_2^2 - X_3^2, \\ Y_3^2 &= X_0^2 - X_1^2 - X_2^2 + X_3^2 \end{aligned}$$

is biholomorphic equivalent to the Satake compactification of  $\mathbb{H}_2/\Gamma'$  for a certain subgroup  $\Gamma' \subset \mathrm{Sp}(2, \mathbb{Z})$ . This variety has 96 singularities which correspond to certain zero-dimensional cusps and these singularities are ordinary double points (nodes). In the paper [CM] it has been pointed out that the results of [GN] imply that a (projective) small resolution of this variety is a rigid Calabi-Yau manifold  $\tilde{\mathcal{X}}$ .

We describe the basic occurring groups: We use the standard notations,  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ :

$$\begin{aligned} \Gamma_n[l] &= \ker(\mathrm{Sp}(n, \mathbb{Z}) \longrightarrow \mathrm{Sp}(n, \mathbb{Z}/l\mathbb{Z})), \\ \Gamma_n[l, 2l] &= \{ M \in \Gamma_n[l]; \quad A^t B/l \text{ and } C^t D/l \text{ have even diagonal} \}, \\ \Gamma_{n,0}[l] &= \{ M \in \Gamma_n; \quad C \equiv 0 \pmod{l} \}, \\ \Gamma_{n,0,\vartheta}[l] &= \{ M \in \Gamma_n; \quad C \equiv 0 \pmod{l}, \quad C^t D/l \text{ has even diagonal} \}. \end{aligned}$$

The group  $\Gamma_{n,0}[l]$  can be extended by the Fricke involution

$$J = J_l = \begin{pmatrix} 0 & E/\sqrt{l} \\ -\sqrt{l}E & 0 \end{pmatrix} \quad \left( JZ = -\frac{1}{lZ} \right).$$

We denote this extension (of index 2) by

$$\hat{\Gamma}_{n,0}[l] = \Gamma_{n,0}[l] \cup J\Gamma_{n,0}[l].$$

The group  $\Gamma'$ , which belongs to van Geemen's and Nygaard's variety is a subgroup of index two of the group

$$\Gamma_2[2, 4] \cap \Gamma_{2,0,\vartheta}[4] = \{M \in \Gamma_2[2, 4]; \quad C \equiv 0 \pmod{4}, \quad C/4 \text{ has even diagonal}\},$$

namely

$$\Gamma' = \{M \in \Gamma_2[2, 4] \cap \Gamma_{2,0,\vartheta}[4]; \quad \det D \equiv \pm 1 \pmod{8}\}.$$

In [FSM] we introduced a certain subgroup  $\Gamma_{2,0}[2]_{\mathbf{n}}$  of index two of  $\Gamma_{2,0}[2]$  as kernel of a certain character  $\chi_{\mathbf{n}}$ . This character is the product of the unique non-trivial character of the full Siegel modular group and the character

$$i^{\alpha+\beta+\gamma}, \quad C^t D = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}, \quad \left(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{2,0}[2]\right).$$

The group  $\Gamma'$  is contained in  $\Gamma_{2,0}[2]_{\mathbf{n}}$  as subgroup of index 6144.

The character  $\chi_{\mathbf{n}}$  extends to a character  $\hat{\chi}_{\mathbf{n}}$  of  $\hat{\Gamma}_{2,0}[2]$  by means of

$$\hat{\chi}_{\mathbf{n}}(J) = 1.$$

We denote its kernel by  $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$ . This is an extension of index two of  $\Gamma_{2,0}[2]_{\mathbf{n}}$ . The group  $\Gamma'$  is contained in  $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$  as subgroup of index 12 288. The main result of this paper is:

**Theorem.** *The Siegel modular threefold, which belongs to a group between  $\Gamma'$  and  $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$ , admits a Calabi-Yau model in the following weak sense: There exists a desingularization in the category of complex spaces of the Satake compactification which admits a holomorphic three-form without zeros and whose first Betti number vanishes.*

It has been pointed out by van Geemen that it is not always possible to get a projective model. We want to come back to the question of projectivity in an extra paper. Here we just mention a result from [CFS] that can be proved with the methods of [FSM]:

**Supplement.** *In the case of a group between  $\Gamma_{2,0}[2]_{\mathbf{n}}$  and  $\Gamma_{2,0}[4] \cap \Gamma_2[2]$  one can get a projective Calabi-Yau manifold.*

We will develop a method to compute the Picard number  $\text{pic}$  (rank of the group of divisor classes) and the Euler number for the intermediate groups. In the projective case they determine the Hodge numbers, the Picard number is  $\text{pic} = h^{11}$  and the Euler number is

$$e = 2(h^{11} - h^{12}).$$

For a small resolution  $\tilde{\mathcal{X}}$  of the variety of van Geemen and Nygaard  $\mathcal{X}$  they are known [GN], [CM],  $\text{pic} = 32$ ,  $e = 64$ . To get it for other groups, one needs the action of the group  $\hat{\Gamma}_{2,0}[2]_n$  on the Picard group of the regular locus of  $\mathcal{X}$ . We will determine this action in section 6. The result of this section allow in principle to compute the numbers for all intermediate groups. There are thousands of conjugacy classes of intermediate groups.

In section 7 we treat some simple examples, namely all subgroups of order two of  $\hat{\Gamma}_{2,0}[2]_n/\Gamma'$ .

In a subsequent paper we will treat some more significative cases.

We are grateful for helpful discussions with Bert van Geemen and also for useful comments to a preliminary version of our paper from Philip Candelas. He brought our attention to the paper [BH] of Borisov and Hua, where other examples of complete intersections of 4 quadrics in  $P^7(\mathbb{C})$  that lead to Calabi-Yau manifolds with big fundamental groups are described.

## 1. The variety of van Geemen and Nygaard

We recall some basic facts about Siegel modular varieties. For details we refer to [Fr1]. The symplectic group  $\text{Sp}(n, \mathbb{R})$  acts on the Siegel upper half plane

$$\mathbb{H}_n := \{Z \in \mathbb{C}^{(n,n)}; \quad Z = {}^t\bar{Z} = X + iY, \quad Y > 0 \text{ (positive definite)}\}$$

by means of the formula  $MZ = (AZ + B)(CZ + D)^{-1}$ . For any subgroup  $\Gamma \subset \text{Sp}(n, \mathbb{R})$  that is commensurable with  $\text{Sp}(n, \mathbb{Z})$ , the quotient  $\mathbb{H}_n/\Gamma$  carries a natural structure as quasi projective algebraic variety. The Satake compactification  $\overline{\mathbb{H}_n/\Gamma}$  is a projective variety which is closely related to Siegel modular forms. The Satake compactification can be identified with the projective variety associated to a graded algebra of modular forms. We recall briefly its definition. A modular form  $f$  of weight  $r/2$ ,  $r \in \mathbb{Z}$ , is a holomorphic function  $f$  on  $\mathbb{H}_n$  with the transformation property

$$f(MZ) = v(M) \sqrt{\det(CZ + D)}^r f(Z)$$

for all  $M \in \Gamma$ . In the case  $n = 1$  a regularity condition at the cusps has to be added. Here  $v(M)$  is system of complex numbers of absolute valued one, called the multiplier system. It has to fulfil an obvious cocycle condition. We denote this space by  $[\Gamma, r/2, v]$ . Fixing some starting weight  $r_0$  and a multiplier system  $v$  for it, we define the ring

$$A(\Gamma) := \bigoplus_{r \in \mathbb{Z}} [\Gamma, rr_0/2, v^r].$$

This turns out to be a finitely generated graded algebra and its associated projective variety  $\text{proj } A(\Gamma)$  can be identified with the Satake compactification. The ring depends on the starting weight and the multiplier system but the associated projective variety does not.

Basic examples of modular forms are given by theta series with respect to characteristics. By definition, a theta characteristic is an element  $m = \begin{pmatrix} a \\ b \end{pmatrix}$  from  $(\mathbb{Z}/2\mathbb{Z})^{2n}$ . Here  $a, b \in (\mathbb{Z}/2\mathbb{Z})^n$  are column vectors. The characteristic is called even if  $\mathfrak{t}b = 0$  and odd otherwise. The group  $\text{Sp}(n, \mathbb{Z}/2\mathbb{Z})$  acts on the set of characteristics by

$$M\{m\} := \mathfrak{t}M^{-1}m + \begin{pmatrix} (A\mathfrak{t}B)_0 \\ (C\mathfrak{t}D)_0 \end{pmatrix}.$$

Here  $S_0$  denotes the column built of the diagonal of a square matrix  $S$ . It is well-known that  $\text{Sp}(n, \mathbb{Z}/2\mathbb{Z})$  acts transitively on the subsets of even and odd characteristics. Recall that to any characteristic the theta function

$$\vartheta[m] = \sum_{g \in \mathbb{Z}^n} e^{\pi i(Z[g+a/2] + \mathfrak{t}b(g+a/2))} \quad (Z[g] = \mathfrak{t}gZg)$$

can be defined. Here we use the identification of  $\mathbb{Z}/2\mathbb{Z}$  with the subset  $\{0, 1\} \subset \mathbb{Z}$ . It vanishes if and only if  $m$  is odd. Recall also that the formula

$$\vartheta[M\{m\}](MZ) = v(M, m) \sqrt{\det(CZ + D)} \vartheta[m](Z)$$

holds for  $M \in \Gamma_n$ , where  $v(M, m)$  is a rather delicate 8<sup>th</sup> root of unity which depends on the choice of the square root. Sometimes we will use the notation

$$\vartheta[m] = \vartheta \begin{bmatrix} a_1 a_2 \\ b_1 b_2 \end{bmatrix} \quad \text{for } m = \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix}.$$

Following van Geemen and Nygaard, we consider the 8 functions

$$\begin{aligned} &\vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix}(Z), \quad \vartheta \begin{bmatrix} 00 \\ 10 \end{bmatrix}(Z), \quad \vartheta \begin{bmatrix} 00 \\ 01 \end{bmatrix}(Z), \quad \vartheta \begin{bmatrix} 00 \\ 11 \end{bmatrix}(Z), \\ &\vartheta \begin{bmatrix} 00 \\ 00 \end{bmatrix}(2Z), \quad \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix}(2Z), \quad \vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix}(2Z), \quad \vartheta \begin{bmatrix} 11 \\ 00 \end{bmatrix}(2Z). \end{aligned}$$

If we denote them by  $Y_0, \dots, Y_3, X_0, \dots, X_3$ , then classical theta relations show that the relations listed at the beginning of the introduction hold. The classical theta transformation formalism shows that the eight forms are modular forms of weight  $1/2$  for the group  $\Gamma'$  introduced in the introduction and that their multipliers on this group agree. Since this is standard, we only give a short sketch of the proof. The theta series  $X_i$  are the so-called theta series of second

kind. One knows classically that they have the same multipliers  $\kappa(M)$  on the group  $\Gamma_2[2, 4]$  (s. for example [Ru]). By conjugation with the transformation  $Z \mapsto 2Z$  one shows that the  $Y_i$  have the same multipliers  $\tilde{\kappa}(M)$  on the group  $\Gamma_{2,0,\vartheta}[2]$ . We have to find the subgroup  $\Gamma'$  of  $\Gamma_2[2, 4] \cap \Gamma_{2,0,\vartheta}[2]$ , where  $\kappa$  and  $\tilde{\kappa}$  agree. This just means that  $\vartheta[0](Z)\vartheta[0](2Z)$  and  $\vartheta[0](2Z)^2$  have the same multipliers. But these are the standard thetas series with respect to the binary forms  $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$  and  $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ . The advantage of binary forms is that they have an even number of variables and standard formula can be used, for example [Fr2], 7.1.  $\square$

We can express this in saying that the  $Y_0, \dots, X_3$  are contained in the ring

$$A(\Gamma') := \bigoplus_{r \in \mathbb{Z}} [\Gamma', r/2, \kappa^r].$$

The precise result, slightly extending results of [GN], states:

**1.1 Proposition.** *Let be*

RingvGN

$$\Gamma' = \{M \in \Gamma_2[2, 4]; \quad C \equiv 0 \pmod{4}, \text{ diag } C \equiv 0 \pmod{8}, \det D \equiv \pm 1 \pmod{8}\}.$$

*The ring  $A(\Gamma')$  is generated by the eight theta series above. The relations*

$$\begin{aligned} Y_0^2 &= X_0^2 + X_1^2 + X_2^2 + X_3^2, \\ Y_1^2 &= X_0^2 - X_1^2 + X_2^2 - X_3^2, \\ Y_2^2 &= X_0^2 + X_1^2 - X_2^2 - X_3^2, \\ Y_3^2 &= X_0^2 - X_1^2 - X_2^2 + X_3^2 \end{aligned}$$

*are defining relations. They define a subvariety  $\mathcal{X}$  of  $P^7(\mathbb{C})$  which can be identified with the Satake compactification of  $\mathbb{H}_2/\Gamma'$ .*

*Proof.* Proposition 2.5 of [GN] says that this complete intersection is the Satake compactification for some subgroup of  $\text{Sp}(2, \mathbb{Z})$ . Necessarily this must be a subgroup of what we called  $\Gamma'$ . An index computation gives that they agree. The equality of the complete intersection and the Satake compactification shows that  $A(\Gamma')$  must be the normalization of the factor ring  $\mathbb{C}[X_0, \dots, Y_3]$  by the ideal generated by the above 4 relations. Using Serre's criterion for normality, it follows that this ring is normal. This proves 1.1.  $\square$

In [GN] the modular form of weight 3

$$T = \vartheta \begin{bmatrix} 10 \\ 00 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 10 \\ 01 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 01 \\ 00 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 01 \\ 10 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 11 \\ 00 \end{bmatrix} (Z) \vartheta \begin{bmatrix} 11 \\ 11 \end{bmatrix} (Z)$$

has been introduced. The differential form

$$\omega = T dz_0 \wedge dz_1 \wedge dz_2, \quad Z = \begin{pmatrix} z_0 & z_1 \\ z_1 & z_2 \end{pmatrix},$$

is invariant under  $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$  (The invariance under  $\Gamma_{2,0}[2]_{\mathbf{n}}$  has been proved in [FSM]. The behavior under the Fricke involution can be derived from the following explicit formula.)

**1.2 Lemma.** *In terms of the coordinates  $X_0, \dots, X_3, Y_0, \dots, Y_3$  we have that,  $\omega$  equals up to a multiplicative constant,  $\omega$  equals*

$$\frac{X_2^4}{Y_0 Y_1 Y_2 Y_3} d(X_0/X_2) \wedge d(X_1/X_2) \wedge d(X_3/X_2).$$

*Proof.* This is essentially the form of  $\omega$ , which has been derived in [FSM] (see Theorem 4.5 and the formulae before it).  $\square$

The zero locus of  $T$  consists of the fixed point sets of all  $M \in \Gamma_{2,0}[2]_n$ , which are conjugate inside  $\mathrm{Sp}(2, \mathbb{Z})$  to the diagonal matrix with diagonal  $(1, -1, 1, -1)$ . It can be checked that all these  $M$  are contained in  $\Gamma'$ . Hence we obtain the following well-known result:

**1.3 Remark.** *The differential form  $\omega$  defines a holomorphic differential form without zeros on the smooth variety  $\mathbb{H}_2/\Gamma'$ . It is invariant under  $\hat{\Gamma}_{2,0}[2]_n$ .*

## 2. Automorphisms of the variety of van Geemen and Nygaard

As we will see (2.1), the group  $\Gamma'$  is normal in  $\hat{\Gamma}_{2,0}[2]$ . Hence this group acts on the variety of van Geemen and Nygaard. We want to describe this action. We will see that this action can be described by a linear action on the variables  $Y_0, \dots, X_3$ , more precisely by a finite subgroup of  $\mathrm{PGL}(8, \mathbb{C})$ . Recall that  $\Gamma_{2,0}[2]$  is generated by the matrices of the form

$$\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}, \quad \begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix}, \quad \begin{pmatrix} E & 0 \\ 2S & E \end{pmatrix} \quad (S = S' \text{ integral}).$$

Let  $M \in \hat{\Gamma}_{2,0}[2]$ . For  $f \in [\Gamma', 1/2, v_\vartheta]$  we set

$$f|M(Z) = \sqrt{\det(CZ + D)}^{-1/2} f(MZ).$$

The map  $f \mapsto f|M$  is an automorphism  $\varphi_M$  of the 8-dimensional space spanned by  $Y_0, \dots, X_3$ . It depends on the choice of a square root of  $\det(CZ + D)$ . Hence  $\pm\varphi_M$  is well defined. Using standard theta transformation formulae we can compute these automorphisms for the generators. It is sufficient to take the following 4.

matrix	corresponding transformation
$\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}, U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$(Y_0, Y_1, Y_3, Y_2, X_0, X_3, X_2, X_1)$
$\begin{pmatrix} {}^tU & 0 \\ 0 & U^{-1} \end{pmatrix}, U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$(Y_0, Y_3, Y_2, Y_1, X_0, X_1, X_3, X_2)$
$\begin{pmatrix} E & 0 \\ S & E \end{pmatrix}, S = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	$(Y_0, -iY_1, Y_2, -iY_3, X_1, X_0, X_3, X_2)$
$J$ (Fricke involution)	$\sqrt{2} \cdot (X_0, X_1, X_2, X_3, Y_0/2, Y_1/2, Y_2/2, Y_3/2)$

**2.1 Lemma.** *The group  $\Gamma'$  is normal in  $\Gamma_{2,0}[2]$ . The group  $\mathcal{G}$  generated by the transformations  $\pm\varphi_M$ ,  $M \in \hat{\Gamma}_{2,0}[2]$ , is already generated by the above four transformations. It has order 98 304. The map  $M \mapsto \pm\varphi_M$  defines a homomorphism*

$$\hat{\Gamma}_{2,0}[2] \longrightarrow \mathcal{G}/\pm.$$

*The group  $\mathcal{G}$  contains the subgroup  $\mathcal{Z}$  of order 4 which is generated by multiplication with  $i$ . The above homomorphism induces an isomorphism*

$$\hat{\Gamma}_{2,0}[2]/\Gamma' \xrightarrow{\sim} \mathcal{G}/\mathcal{Z} \quad (\text{order } 24\,576).$$

*Proof.* Since the generators of  $\Gamma_{2,0}[2]$  act on  $\mathcal{X}$ , the group  $\Gamma'$  must be a normal subgroup. The rest follows by comparing indices.  $\square$

### 3. The stabilizer of a node

The variety  $\mathcal{X}$  has 96 singularities, which all are ordinary double points (nodes). They are zero dimensional boundary points, but not each zero dimensional boundary is singular. In coordinates the singularities can be described as follows:

One node is given by

$$P = [\sqrt{2}, 0, \sqrt{2}, 0, 1, 1, 0, 0].$$

Changing signs it produces 8 nodes, which are characterized by the property  $Y_1 = Y_3 = X_2 = X_3 = 0$ . Similarly, one gets 8 nodes with  $Y_1 = Y_3 = X_0 = X_1 = 0$ . So one has 16 nodes with  $Y_1 = Y_3 = 0$ . In the same way one gets 16 nodes with the property  $Y_i = Y_j = 0$  for each other pair  $0 \leq i < j \leq 3$ . This gives  $96 = 8 \cdot 16$  nodes. It is easy to check by hand that they exhaust all singular points. This description of the nodes also shows:

**3.1 Lemma.** *The group  $\hat{\Gamma}_{2,0}[2]$  acts on the 96 nodes is transitive.*

ActNode

The following matrices

$$\begin{aligned} M_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, & M_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ M_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & M_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ M_5 &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & M_6 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \circ J \end{aligned}$$

are contained in  $\hat{\Gamma}_{2,0}[2]$  and stabilize the node  $P$ . We consider the group, which is generated by them and  $\Gamma'$ . One can check that the factor group mod  $\Gamma'$  has order 128. Together with 3.1 we obtain:

**3.2 Proposition.** *The stabilizer  $\hat{\Gamma}_{2,0}[2]_P$  of the node  $P$  inside  $\hat{\Gamma}_{2,0}[2]$  is generated by  $\Gamma'$  and the matrices  $M_1, M_2, \dots, M_6$ .*

StabGen

In a neighborhood of  $P$  we can use the affine coordinates

$$\eta_0 = Y_0/X_1, \eta_1 = Y_1/X_1, \eta_2 = Y_2/X_1, \eta_3 = Y_3/X_1,$$

$$\xi_0 = X_0/X_1, \xi_2 = X_2/X_1, \xi_3 = X_3/X_1$$

Then substituting the affine version of the third equation in the fourth we get

$$\eta_1^2 - \eta_3^2 = 2(\xi_2^2 - \xi_3^2)$$

Setting

$$x_1 = \eta_1 - \eta_3, \quad x_4 = \eta_1 + \eta_3, \quad x_2 = \sqrt{2}(\xi_2 - \xi_3), \quad x_3 = \sqrt{2}(\xi_2 + \xi_3),$$

the relation gets the simple form

$$x_1 x_4 = x_2 x_3.$$

So we have lead to consider the quadric

$$Q := \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4; \quad x_1 x_4 = x_2 x_3\}.$$

This is a three dimensional affine variety with a unique singularity at the origin. The above construction gives an étale map of germs

$$(\mathcal{X}, P) \longrightarrow (Q, 0).$$



Sometimes we write the elements of  $\mathbb{C}^4$  as matrices

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Then  $Q$  is defined by  $\det X = 0$ . The group  $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})$  acts on the quadric by means of

$$X \mapsto AX {}^tB.$$

In this way we can consider  $(\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C}))/\mathbb{C}^*$  as subgroup of  $\mathrm{GL}(4, \mathbb{C})$ . Another transformation, which leaves the quadric invariant, is  $X \mapsto {}^tX$ . We also can consider it as element of  $\mathrm{GL}(4, \mathbb{C})$ .

We consider the transformations

$$\begin{aligned} m_1 \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} &= \begin{pmatrix} x_4 & x_2 \\ x_3 & x_1 \end{pmatrix}, & m_2 \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} &= \begin{pmatrix} -ix_1 & -x_2 \\ x_3 & -ix_4 \end{pmatrix}, \\ m_3 \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} &= \begin{pmatrix} -x_1 & ix_2 \\ ix_3 & x_4 \end{pmatrix}, & m_4 \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} &= \begin{pmatrix} -x_1 & -x_2 \\ x_3 & x_4 \end{pmatrix}, \\ m_5 \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} &= \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}, & m_6 \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} &= \begin{pmatrix} x_2 & x_1 \\ x_4 & x_3 \end{pmatrix}. \end{aligned}$$

They are contained in the extension of index two of the image of the group  $(\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C}))$ , which is generated by  $X \mapsto {}^tX$ . Hence this subgroup acts on  $Q$ .

**3.3 Lemma.** *The transformations*

DefGH

$$m_1, m_2, m_3, m_4, m_5, m_6$$

generate a group  $G$  of order 256.

Since the proofs of this Lemma and the following Proposition can be given by computation, we omit them.

**3.4 Proposition.** *The assignment*

IsoStabs

$$M_i \mapsto m_i \quad (1 \leq i \leq 6)$$

induces an isomorphism

$$\hat{\Gamma}_{2,0}[2]_P/\Gamma' \xrightarrow{\sim} G.$$

*The described identification of germs  $(X(\Gamma'), P)$  and  $(Q, 0)$  is equivariant.*

We are interested in the subgroup of index two  $\hat{\Gamma}_{2,0}[2]_{\mathfrak{n}}$ . We have to intersect it with the stabilizer. It is easy to check that the elements

$$M_2^2, M_3^2, M_2M_1, M_3M_1, M_4M_1, M_5M_1, M_6M_1$$

are contained in  $\hat{\Gamma}_{2,0}[2]_{\mathfrak{n}}$ . One also can check that the elements

$$m_2^2, m_3^2, m_2m_1, m_3m_1, m_4m_1, m_5m_1, m_6m_1.$$

generate a group  $H$  of order 128. In this way one obtains:

**3.5 Proposition.** *The stabilizer of  $P$  in  $\hat{\Gamma}_{2,0}[2]_{\mathfrak{n}}$  is a subgroup of index two of  $\hat{\Gamma}_{2,0}[2]_P$ . The restriction of 3.4 induces an isomorphism* Isozstabs

$$(\hat{\Gamma}_{2,0}[2]_P \cap \hat{\Gamma}_{2,0}[2]_{\mathfrak{n}})/\Gamma' \xrightarrow{\sim} H.$$

## 4. Some results about crepant resolutions

In this section we recall some known result about projective crepant resolutions.

We need the notion of a crepant resolution for certain normal three dimensional varieties  $X$ .

**4.1 Definition.** *Let  $X$  be a connected three dimensional normal complex space with singular locus  $S$ . Assume that for each point  $a \in X$  there exists an open neighborhood  $U$  and holomorphic three form  $\alpha$  on  $U - S$  without zeros. A **crepant resolution**  $f : \tilde{X} \rightarrow X$  is a holomorphic map of a connected (smooth) complex manifold  $\tilde{X}$  onto  $X$ , such that  $\tilde{X} - f^{-1}(S) \rightarrow X - S$  is biholomorphic and such that  $\alpha$  extends to a holomorphic three form without zeros on the inverse image  $\tilde{U} = f^{-1}(U)$ .* DefCrep

The existence of a *quasiprojective* crepant desingularization is only a local question (in the three-dimensional case):

**4.2 Lemma (Roan).** *Under the assumptions of 4.1 the following holds: The existence of a crepant desingularization is granted, if there exists an open covering  $U_i \subset X$ , such that each  $U_i$  admits a crepant desingularization.* LocCrep

We reproduce the argument of Roan: The singular locus is a curve  $S$ . Over the generic point of  $S$  the crepant resolution is unique. For this reason one can choose the crepant resolutions over the finitely many singular points of  $S$  arbitrarily and glue them to a global resolution.

We use a result of a general result about the existence of a resolutions of quotient singularities:

**4.3 Theorem.** *Let  $X$  be a complex manifold of dimension three and  $G$  a finite group of automorphisms of it. Assume that every point of  $X/G$  admits an open neighborhood (in the analytic topology) such that on its regular locus there exists a three-form without zeros. Then  $X/G$  admits a crepant desingularization.* Ito

Because of 4.2 it is sufficient to prove this result in the situation  $\mathbb{C}^3/G$  where  $G \subset \mathrm{GL}(3, \mathbb{C})$  is a finite subgroup. We can assume that  $G$  doesn't contain quasi reflections. Then the assumption in 4.3 gives  $G \subset \mathrm{SL}(3, \mathbb{C})$ . We refer to [Re] (especially section 5) for historical comments and basic results.

## 5. Some quotients of an ordinary double point

We will study the node  $(Q, 0)$  and some of its quotients. This is related to work of Davis [Da], where also several quotients have been considered.

**5.1 Lemma.** *The restriction of*

**Diffinv**

$$\alpha = \frac{1}{x_1^2 - x_4^2} (x_1 dx_1 + x_4 dx_4) \wedge dx_2 \wedge dx_3$$

*is a holomorphic differential form of degree three on  $Q - \{0\}$  without zeros. If one identifies a small neighborhood of the origin in  $Q$  with a small neighborhood of  $P \in X(\Gamma')$ , we have  $\omega = h\alpha$ , where  $h$  is a holomorphic invertible function on this neighborhood.*

*Proof.* We cover  $Q$  by 4 charts corresponding to  $x_i \neq 0$ . For example the part  $x_1 \neq 0$  is the image of

$$\{(x_1, x_2, x_3) \in \mathbb{C}^3; x_1 \neq 0\} \longrightarrow \mathbb{C}^4, \quad (x_1, x_2, x_3) \longmapsto (x_1, x_2, x_3, x_2 x_3 / x_1).$$

Pulling back  $\alpha$  we get

$$\frac{1}{x_1^2 - (x_2 x_3 / x_1)^2} (x_1 dx_1 - x_2^2 x_3^2 / x_1^3 dx_1) \wedge dx_2 \wedge dx_3 = \frac{dx_1 \wedge dx_2 \wedge dx_3}{x_1}$$

This is holomorphic and without zeros on this chart. The other charts are treated in a similar way.

Since  $\alpha$  and  $\omega$  both are 3-forms without zeros outside the singularity, we get  $\omega = h\alpha$ , where  $h$  is a holomorphic function without zeros outside the singularity. Since isolated singularities of analytic functions in more than one variable cannot exist,  $h$  and  $h^{-1}$  are holomorphic also at the singularity.

□

The following result can be found in [Fri], see also [Jo], 6.3.

**5.2 Proposition.** *The quadric  $Q$  admits a small desingularization  $\tilde{Q} \rightarrow Q$ . This means that  $\tilde{Q}$  is a smooth connected variety, the inverse image of the node 0 is a curve and the map from the complement of this curve maps biholomorphically onto  $Q - \{0\}$ . Such a desingularization is crepant.*

A small resolution is not unique. Actually there exist two different isomorphy classes of such small resolutions. They can be obtained by blowing up the ideals  $(x_1, x_3)$  or  $(x_1, x_4)$  in  $\mathbb{C}[x_1, x_2, x_3, x_4]/(x_1 x_4 - x_2 x_3)$ . From this explicit description one can derive:

**5.3 Lemma.** *The elements of the image of  $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})$  extend to biholomorphic transformations of any small resolution, but the transformation* Fort

$$X \mapsto {}^tX,$$

*which is also an automorphism of  $Q$ , does not.*

The group  $G$  (see 3.4) is not contained in the image of  $\mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})$ . The intersection with this group defines a subgroup  $G_0 \subset G$  of index two. It is generated by the elements  $m_1 m_5, m_2, m_3, m_4, m_6$ . One checks that these elements have determinant 1 (considered in  $\mathrm{GL}(4, \mathbb{C})$ ). Hence  $G_0$  also can be defined as intersection of  $G$  with  $\mathrm{SL}(4, \mathbb{C})$ . We denote by  $H_0$  the intersection of  $H$  and  $G_0$ . This is a group of order 64.

**5.4 Lemma.** *The groups  $H$  and  $H_0$  have the same center. It is the group of order 2 generated by the transformation  $x \mapsto -x$ .* Cent

We omit the proof, since it can be done by simple computation.

**5.5 Lemma.** *The differential form  $\alpha$  on  $\mathbb{C}^4$  (s. 5.1) is invariant under  $H$ . Hence also the function  $h$  in 5.1 is  $H$ -invariant.* Difinv

*Proof.* The invariance can be checked directly for the generators. □

**5.6 Theorem.** *Let  $K$  be any subgroup of  $H$ . Then the quotient  $Q_0 := Q/K$  admits a crepant desingularization.* Crep

For the proof we have to differ between 4 types of subgroups  $K$ .

- 1)  $K$  is contained in the subgroup  $H_0$ .
- 2)  $K$  contains the transformation  $x \mapsto -x$ .
- 3)  $K$  contains one of the two transformations

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mapsto \begin{pmatrix} -x_1 & x_2 \\ x_3 & -x_4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_1 & -x_2 \\ -x_3 & x_4 \end{pmatrix}$$

in its center.

- 4)  $K$  is a the cyclic subgroup of order 4 that is conjugated to one of the two following (given by a generator of order 4):

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_2 & x_4 \\ x_1 & x_3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_2 & ix_4 \\ -ix_1 & x_3 \end{pmatrix}.$$

These classes are not disjoint. But each subgroup of  $H$  is contained in at least one of the 4 classes. This can be checked by hand or quicker by means of a computer.

We are going to discuss the 4 cases. We begin with the

*First type.* Because of 5.3 in this case the action of  $K$  extends to a small resolution  $\tilde{Q} \rightarrow Q$ . The differential form  $\alpha$  extends to a holomorphic differential form without zeros on  $\tilde{Q}$ , since singularities or zeros can only occur in codimension one on a smooth variety. By 4.3 the variety  $\tilde{Q}/K$  admits a crepant resolution. Hence  $Q/K$  also admits one.

*Second type.* We start to blow up the origin of  $\mathbb{C}^4$ . The group  $K$  still acts biholomorphically on this blow up. A typical chart of the blow up is the  $\mathbb{C}^4$  with the coordinates

$$(t_1, t_2, t_3, x_4) = (x_1/x_4, x_2/x_4, x_3/x_4, x_4).$$

We consider in the blow up of  $\mathbb{C}^4$  the closed smooth subvariety  $\tilde{Q}$ , which is defined in this chart by  $t_1 = t_2 t_3$ . Its image in  $\mathbb{C}^4$  is  $Q$ . Hence  $\tilde{Q} \rightarrow Q$  is just a desingularization of  $Q$ . (Actually it is the blow-up of  $Q$  at the origin.) The chart of  $\tilde{Q}$ , which we consider, is a  $\mathbb{C}^3$  with the coordinates  $t_2, t_3, x_4$ . The differential form  $\alpha$  in these coordinates can be computed. Up to a constant factor it is

$$x_4 dt_2 \wedge dt_3 \wedge dx_4.$$

So it gets a zero of order one along  $x_4 = 0$ . The transformation  $x \rightarrow -x$  just changes the sign of each variable  $x_i$ . Hence it acts on  $t_2, t_3, x_4$  as reflection, which changes the sign of the third variable only. The quotient is a  $\mathbb{C}^3$  with the coordinates

$$(t_2, t_3, t_4) = (x_2/x_4, x_3/x_4, x_4^2).$$

Hence  $\tilde{Q}/\pm$  is a smooth variety, the affine piece in consideration a  $\mathbb{C}^3$  with coordinates  $t_2, t_3, t_4$ . In these coordinates  $\alpha$  appears as holomorphic differential form without zeros. By the general theorem 4.3, the quotient  $\tilde{Q}/K$  and hence  $Q/K$  admits a crepant resolution.

*Third type.* The two cases are equivalent, hence we can assume that

$$\sigma(x_1, x_2, x_3, x_4) = (x_1, -x_2, -x_3, x_4)$$

is in the center of  $K$ . The ideal  $(x_2, x_3)$  is invariant under  $K$ , since it describes the fixed point locus of  $\sigma$  which is in the center of  $K$ . Hence the action of  $K$  extends to an action by biholomorphic transformations on the blow up  $\mathcal{C}$  of  $\mathbb{C}^4$  along this ideal. The manifold  $\mathcal{C}$  can be covered by two  $\mathbb{C}^4$  using the coordinates  $(x_1, x_2/x_3, x_3, x_4)$  and  $(x_1, x_2, x_3/x_2, x_4)$ . We take the quotient of  $\mathcal{C}$  by  $\sigma$ . The quotient  $\mathcal{C}/\sigma$  is covered by two  $\mathbb{C}^4$  with coordinates

$$(u_1, u_2, u_3, u_4) = (x_1, x_2/x_3, x_3^2, x_4) \quad \text{and} \quad (v_1, v_2, v_3, v_4) = (x_1, x_2^2, x_3/x_2, x_4).$$

We consider the subvariety  $Q' \subset \mathcal{C}/\sigma$ , which in the two affine pieces is given by  $u_1 u_4 = u_2 u_3$  and  $v_1 v_4 = v_2 v_3$ . This variety has two singular points, which correspond to the origins of the affine pieces and which are ordinary double

points. There is a natural projection  $Q' \rightarrow Q/\sigma$ . The group  $K$  acts on  $Q'$ . We need the stabilizers of the two singularities.

*Claim.* The stabilizer of the singularity  $u_1 = \dots = u_4 = 0$  acts by substitutions of the form

$$U \mapsto CU^tD, \quad \text{where} \quad U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}.$$

*Proof of the Claim.* We consider an element of the stabilizer. It might be of the form

$$X \mapsto X \mapsto AX^tB \quad \text{or} \quad X \mapsto A^tX^tB.$$

We have to use now that this transformation commutes with  $\sigma$ . In each case this means that  $A$  and  $B$  both are of the form  $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$  or both are of the form  $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ . Both cases are similar. For simplicity we take the case  $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$  and  $B = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ . Then the transformation  $AX^tB$  corresponds to  $CU^tD$ , where

$$C = \begin{pmatrix} 1 & 0 \\ 0 & d^2\alpha/a \end{pmatrix}, \quad D = \begin{pmatrix} a\alpha & 0 \\ 0 & a\delta/d\alpha \end{pmatrix}.$$

But the transformation  $A^tX^tB$  interchanges the  $u$ - and  $v$ -chart and especially the two singularities are interchanged. Hence this transformation is not contained in the stabilizer.

We consider now the holomorphic map

$$Q'/\mathcal{K} \rightarrow Q/\mathcal{K},$$

which is induced by  $Q' \rightarrow Q/\sigma$ . The differential form  $\alpha$  can be considered as meromorphic differential form on  $\mathcal{C}/\sigma$ . One checks that in the coordinates  $u_1, u_2, u_3, u_4$  it is given by the same equation as the original  $\alpha$ , just replacing the letters  $x$  by  $u$ . The same is true for the coordinates  $v_1, v_2, v_3, v_4$ . Hence  $\alpha$  gives a holomorphic differential form without zeros on the regular locus of  $Q'$ . The claim shows that  $\mathcal{K}$  extends to a suitable chosen small resolution  $\tilde{Q}'$  of  $Q'$ . To be concrete one can blow up the irreducible surface, which is defined in the  $u$ -coordinates by the ideal  $(u_2, u_3)$  and in the  $v$ -coordinates by  $(v_2, v_3)$ . Then the group  $K$  acts on  $\tilde{Q}'$ . The differential form  $\alpha$  extends to a holomorphic differential form on  $\tilde{Q}'$  without zeros and is invariant under  $\mathcal{K}$ .

*Fourth type.* We consider the first case,  $\sigma(x_1, x_2, x_3, x_4) = (x_2, x_4, x_1, x_3)$ , the second is similar. It is better then to use the coordinates

$$y_1 = x_1 + x_4, \quad y_2 = x_1 - x_4, \quad y_3 = x_2 + x_3, \quad y_4 = x_2 - x_3.$$

The quadric then takes the equation  $y_1^2 + y_4^2 = y_2^2 + y_3^2$ . We blow up the ideal  $(y_2, y_4)$ . We denote by  $\mathcal{C}$  the blow up of  $\mathbb{C}^4$  along this ideal. One chart of the blow up is  $(y_1, y_2, y_3, y_4/y_2)$ . Taking quotient by  $\sigma^2$  gives the chart

$$(u_1, u_2, u_3, u_4) = (y_1, y_2^2, y_3, y_4/y_2).$$

Now the quadric  $y_1^2 + y_4^2 = y_2^2 + y_3^2$  gets the form

$$u_1^2 + u_2(u_4^2 - 1) - u_3^2 = 0.$$

This 3-fold has two singular points,  $(0, 0, 0, 1)$  and  $(0, 0, 0, -1)$ . Take the first one. For this point one can take as local parameter

$$v_4 := u_4^2 - 1 \quad (= (u_4 - 1)(u_4 + 1)).$$

Now the 3-fold is given by  $u_1^2 + u_2v_4 - u_3^2 = 0$ . Hence the singularity is an ordinary double point. This consideration shows that the transform of the quadric appears in  $\mathcal{C}/\sigma^2$  as 3-fold with two singular points, which are nodes. The transformation  $\sigma$  interchanges the two nodes and hence acts without fixed points. The rest of the proof is analogously to the third case.

Now the general result 4.3 shows that  $\tilde{Q}'/\mathcal{K}$  admits a crepant resolution. This gives a crepant resolution of  $Q/\mathcal{K}$ .  $\square$

We recall that we defined a local etale map

$$(X(\Gamma'), P) = (\mathcal{X}, P) \longrightarrow (Q, 0).$$

If  $\Gamma$  is a group between  $\Gamma'$  and  $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$  and  $\mathcal{K}$  the corresponding subgroup of  $\mathcal{H}$ , we still get a local etale map

$$(X(\Gamma), P) \longrightarrow (Q/\mathcal{K}, 0).$$

Pulling back a crepant resolution of  $Q/K$  we get a crepant resolution of some affine neighborhood of  $P$  in  $X(\Gamma)$ .

Now we can prove the a basic result of this paper, formulated in the introduction, which states that the Siegel threefold for each group between  $\Gamma'$  and  $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$  admits a (weak) Calabi-Yau model: By 4.3 there exists a crepant resolution of the complement of the (images of the) nodes in  $X(\Gamma)$ . As we have just seen for each node there exists an open neighborhood which admits a crepant resolution. Hence we can apply 4.2 to obtain a (not necessarily projective) crepant resolution of  $X(\Gamma)$ .

It is a natural question whether a group  $\Gamma$  extends to a group of biholomorphic maps of a crepant resolution  $\tilde{\mathcal{X}}$ . Such a resolution is not unique. There exists a projective one but there also exist some which are not projective. A necessary condition of  $\Gamma$  to extend is that the stabilizers of the nodes extend as described in 5.3. This is a condition which can be checked. Assume that it is satisfied. Then we can choose one node  $a$  and desingularization of this node. We can extend  $\Gamma_a$  to this desingularization. Let now  $g \in \Gamma$ . The choice of the resolution of  $a$  dictates us the choice of the resolution at  $g(a)$  and the assumption about  $G_a$  makes this choice independent of the choice of  $g$ . In this way we resolve the whole orbit of  $a$  and then in the same way the other orbits. In this way we obtain:

**5.7 Lemma.** *Let  $\Gamma$  be a group between  $\Gamma'$  and  $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$ . Assume that the stabilizer at an arbitrary node satisfies the local condition 5.3. Then there exists a crepant resolution  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$  in the category of complex spaces such that  $\Gamma$  extends biholomorphically to  $\tilde{\mathcal{X}}$ .* ExtCom

As van Geemen pointed out to the authors it is usually not possible to get  $\tilde{\mathcal{X}}$  as a projective variety. For example one can show that there are freely acting groups of order  $\#\Gamma/\Gamma' = 32$ . For them it is not possible to get a projective  $\tilde{\mathcal{X}}$ .

## 6. The Picard group

In the following we consider a group  $\Gamma$  between  $\Gamma'$  and  $\hat{\Gamma}_{2,0}[2]_{\mathbf{n}}$ . We denote by  $X(\Gamma)$  the Satake compactification of  $\mathbb{H}_2/\Gamma$  and by  $X(\Gamma)_{\text{reg}}$  its regular locus. We want to compute the Picard number of this variety. We denote it by

$$\text{pic}(\Gamma) := \dim \text{Pic}(X(\Gamma)_{\text{reg}}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Since the map  $X(\Gamma) \rightarrow X(\Gamma')$  is unramified in codimension one, we have

$$\text{Pic}(X(\Gamma)_{\text{reg}}) = \text{Pic}(X(\Gamma')_{\text{reg}})^{\Gamma}.$$

in [GN] the Picard number has been computed for  $\Gamma'$ :

$$\text{pic}(\Gamma') = 32.$$

The proof of van Geemen and Nygaard rests on counting numbers of points with values in some finite fields and making use of the Weil conjectures. Therefore it may be difficult to get from these computations the action of  $\hat{\Gamma}_{2,0}[2]$  on  $\text{Pic}(\mathcal{X}_{\text{reg}})$ . For this reason we need some explicit description of generating divisors.

We use Igusa's cusp form  $\chi_{35}$  of weight 35 for the full Siegel modular group.

**6.1 Theorem.** *The group  $\text{Pic}(\mathcal{X}_{\text{reg}})$  is generated by the irreducible components of the zero divisor of  $\chi_{35}$ .* picGN

The proof needs some computer calculation, which rests on the following informations about the structure of  $\chi_{35}$ . First we recall that  $\chi_{35}$  is the product

$$\chi_{35} = \chi_5 \cdot \chi_{30}$$

of two forms of weight 5 and 30, also for the full modular group but with respect to its non-trivial character. The form  $\chi_5$  can be defined as the product of the ten theta constants. They can be produced as follows: One starts with the most trivial theta constant

$$\vartheta[0](Z) = \sum_{g \in \mathbb{Z}^2} e^{\pi i Z[g]}$$

and applies the full modular group to it. This gives 10 modular forms and there product is  $\chi_5$  (up to a constant factor). The form  $\chi_{30}$  can be constructed in a similar way.



**6.2 Lemma.** *If one applies the full modular group to  $X_0 = \vartheta[0](2Z)$ , one gets (up to constant factors) 60 modular forms (living on  $\Gamma_2[2, 4]$  all with the same multipliers). Their product up to a constant is  $\chi_{30}$ . Examples of forms in the orbit are*

$$\vartheta[0](Z/2) \quad \text{and} \quad X_0, X_1, X_2, X_3.$$

*The 60 modular forms can be written as linear combinations of  $X_0, \dots, X_3$ .*

The last statement is true, since the full modular group acts on the space generated by  $X_0, \dots, X_3$ . This action has been studied in detail by Runge [Ru].

So far we have seen that the zero locus of  $\chi_{35}$  considered on  $\mathcal{X}$  splits into the sum of 70 pairwise different divisors. But these 70 divisors need not to be irreducible.

Now computer algebra comes into the game. Since we know the equations of the 70 divisors in  $P^7(\mathbb{C})$  we can decompose them into irreducibles by using the facility of computer algebra to compute the primary decomposition of an ideal. We have to be a little careful, since computer algebra works well not over  $\mathbb{C}$  but only over a finitely generated field. Hence we have to use a number field  $K$ . We use  $K = \mathbb{Q}(\zeta_8)$ , where  $\zeta_8$  is a primitive 8<sup>th</sup> root of unity. We got the following result using MAGMA [BMP]:

**6.3 Proposition.** *We consider  $\mathcal{X}$  as variety over  $K = \mathbb{Q}(\zeta_8)$  (using the equations of van Geemen and Nygaard). The zero locus of  $\chi_{35}$  is defined over this field. It splits over  $K$  into 132 irreducible components. More precisely we have:*

- 1) *The theta constants  $Y_0, Y_1, Y_2, Y_3$  have irreducible divisors. The other 6 decompose into two irreducibles. Hence  $\chi_5$  contributes with  $16 = 4 + 2 \cdot 6$  irreducible components.*
- 2) *The forms  $X_0, X_1, X_2, X_3$  have irreducible divisors. The other 56 factors of  $\chi_{30}$  decompose into pairs of irreducibles. Hence  $\chi_{30}$  contributes with  $116 = 4 + 2 \cdot 56$  irreducible components.*

We conjecture that these 132 components are irreducible over  $\mathbb{C}$ . But there is no need for us to check this.

The proof of 6.1 now can be given as follows. Using Poincarè duality it is sufficient to construct a system of curves  $C_1, \dots, C_m$ , which are complete and contained in the regular locus of  $\mathcal{X}$  (i.e. they don't contain one of the 96 nodes) and such that the intersection matrix between the 132 divisors and these curves has rank 32. The construction of these curves can be given (again by means of a computer) as follows: Take pairwise intersections of the 132 divisors, decompose them into irreducibles and single out those components, which don't contain nodes.

In this way 6.1 can be proved. This explicit description of the Picard group allows us to describe the action of the group  $\hat{\Gamma}_{2,0}[2]$  on it. The group  $\mathcal{G}$  acts

in a natural way on the ring  $\mathbb{C}[Y_0, \dots, Y_3, X_0, \dots, X_4]$  and on its ideals. Hence we can describe the action of  $\mathcal{G}$  on  $\text{Pic}(\mathcal{X}_{\text{reg}})$  explicitly. Of course  $\mathcal{Z}$  acts trivially. Using the isomorphism  $\hat{\Gamma}_{2,0}[2]/\Gamma' \cong \mathcal{G}/\mathcal{Z}$  we get the action of  $\hat{\Gamma}_{2,0}[2]$  on  $\text{Pic}(\mathcal{X}_{\text{reg}})$ . Using the character table for  $\mathcal{G}/\mathcal{Z}$  which can be produced by computer algebra we get the decomposition into irreducibles.

**6.4 Theorem.** *The space  $\text{Pic}(\mathcal{X}_{\text{reg}}) \otimes_{\mathbb{Z}} \mathbb{Q}$  decomposes under  $\hat{\Gamma}_{2,0}[2]$  into four irreducible components of dimensions 1, 3, 12, 16.* DecIrr

These numbers can be explained as follows:

- 1) The 1-dimensional component comes from the divisor of a modular form.
- 2) The 3-dimensional space comes from the components of the 6 theta constants, which are different from  $Y_0, \dots, Y_3$ .
- 3) The 56 factors of  $\chi_{30}$ , which are different from  $X_0, \dots, X_3$  decompose under  $\Gamma_{2,0}[2]$  into two orbits of 24 and 32 elements. Their irreducible components produce the spaces of dimension 12 and 16.

This explicit picture of the action of  $\hat{\Gamma}_{2,0}[2]$  on  $\text{Pic}(\mathcal{X}_{\text{reg}})$  allows to compute the number  $\text{pic}(\Gamma)$  for every group  $\Gamma$  between  $\Gamma'$  and  $\hat{\Gamma}_{2,0}[2]$  and this can be done by means of a program.

## 7. Involutions

As we have seen, the group  $\hat{\Gamma}_{2,0}[2]/\Gamma' \cong \mathcal{G}/\mathcal{Z}$  has order 24 576. We are interested in its subgroups of order two. One can compute that there are 18 conjugacy classes of such subgroups, and one can show that 10 of them are in the image of  $\hat{\Gamma}_{2,0}[2]_{\text{n}}$ . In the following we list them. There are two possibilities to define such a group. We could describe it by an element  $M \in \hat{\Gamma}_{2,0}[2]$ , such that the image of  $M$  in  $\hat{\Gamma}_{2,0}[2]/\Gamma'$  generates the group. In the case  $M \in \Gamma_{2,0}[2]$  it is enough to consider its image in  $\text{Sp}(2, \mathbb{Z}/8\mathbb{Z})$ , since  $\Gamma'$  contains  $\Gamma_2[8]$ . The other possibility is to give a matrix  $g \in \mathcal{G}$  such that its image in  $\mathcal{G}/\mathcal{Z}$  generates the subgroup of  $\mathcal{G}/\mathcal{Z}$ . Such a  $g$  is determined up to a power of  $i$ . We give the 10 groups the names  $\mathcal{G}2_i$ ,  $1 \leq i \leq 10$ .

$$\begin{array}{ll} \text{Group } \mathcal{G}2_1 & \begin{pmatrix} 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & (Y_0, Y_1, Y_2, Y_3, -X_0, -X_1, -X_2, -X_3) \\ \\ \text{Group } \mathcal{G}2_2 & \begin{pmatrix} 5 & 2 & 6 & 2 \\ 2 & 1 & 2 & 6 \\ 4 & 4 & 1 & 6 \\ 4 & 4 & 6 & 5 \end{pmatrix} & (Y_0, -Y_1, -Y_2, Y_3, X_0, -X_1, -X_2, X_3) \end{array}$$

Group $\mathcal{G}2\_3$	$\begin{pmatrix} 1 & 0 & 2 & 6 \\ 2 & 1 & 2 & 6 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$(Y_0, Y_1, Y_2, Y_3, X_0, -X_1, -X_2, X_3)$
Group $\mathcal{G}2\_4$	$\begin{pmatrix} 3 & 6 & 4 & 2 \\ 4 & 7 & 6 & 2 \\ 0 & 0 & 3 & 4 \\ 0 & 4 & 2 & 7 \end{pmatrix}$	$(-Y_0, -Y_1, Y_2, Y_3, X_0, X_1, -X_2, -X_3)$
Group $\mathcal{G}2\_5$	$\begin{pmatrix} 3 & 2 & 6 & 7 \\ 2 & 3 & 7 & 2 \\ 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \end{pmatrix}$	$(-Y_0, -Y_1, -Y_2, Y_3, X_0, X_1, X_2, -X_3)$
Group $\mathcal{G}2\_6$	$\begin{pmatrix} 5 & 2 & 6 & 7 \\ 0 & 3 & 1 & 0 \\ 0 & 2 & 3 & 0 \\ 6 & 4 & 6 & 5 \end{pmatrix}$	$(-Y_0, -Y_1, -Y_2, Y_3, -X_0, -X_1, -X_2, X_3)$
Group $\mathcal{G}2\_7$	$\begin{pmatrix} 7 & 4 & 7 & 6 \\ 0 & 7 & 2 & 3 \\ 6 & 4 & 5 & 0 \\ 4 & 6 & 4 & 5 \end{pmatrix}$	$(-Y_3, -iY_2, -iY_1, Y_0, X_3, iX_2, iX_1, -X_0)$
Group $\mathcal{G}2\_8$	$\begin{pmatrix} 3 & 7 & 5 & 2 \\ 6 & 1 & 2 & 6 \\ 0 & 6 & 1 & 6 \\ 2 & 0 & 7 & 3 \end{pmatrix}$	$(-Y_1, Y_0, -iY_2, -iY_3, X_2, -iX_1, -X_0, iX_3)$
Group $\mathcal{G}2\_9$	$\begin{pmatrix} 1 & 0 & 7 & 0 \\ 0 & 7 & 0 & 0 \\ 2 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$	$(-iY_1, Y_0, -iY_3, Y_2, X_1, -iX_0, X_3, -iX_2)$
Group $\mathcal{G}2\_10$	Fricke involution	$\sqrt{2} \cdot (X_0, X_1, X_2, X_3, Y_0/2, Y_1/2, Y_2/2, Y_3/2)$

**7.1 Proposition.** *The following table gives the numbers  $\dim \text{Pic}(X(\Gamma)_{\text{reg}}) \otimes_{\mathbb{Z}} \mathbb{Q}$  for the groups  $\Gamma$  corresponding to the groups  $\mathcal{G}2\_i$ ,  $1 \leq i \leq 10$  and the dimension of the fixed point locus of the generating involution in each case. (Dimension -1 means that the locus is empty.)*

Picard number	16	24	16	16	16	20	20	16	16	18
Dimension	-1	0	1	-1	-1	1	1	1	1	1

We want to compute the Picard- and Euler number for a crepant resolution. The numbers of a small resolution  $\tilde{\mathcal{X}}$  of  $\mathcal{X}$  are [GN], [CM]

$$e = 64, \quad \text{pic} = 32.$$

Hence  $\tilde{\mathcal{X}}$  is a rigid Calabi-Yau manifold.

**7.2 Lemma.** *The action of the groups  $\mathcal{G}2\_i$  extends to a suitable crepant desingularization  $\tilde{\mathcal{X}}$ .* ActExt

The proof rests on Lemma . We omit details. □

Each of the groups  $\mathcal{G}2\_i$  is generated by an involution  $\sigma_i$ . We need information by the fixed point locus. It can be checked that there is no component of dimension two. We also know that the Calabi-Yau 3-form is invariant under  $\sigma_i$ . This implies that the action of  $\sigma_i$  on the tangent space of a fixed point can be diagonalized with diagonal  $(-1, -1, 1)$ . From this follows:

**7.3 Lemma.** *The fixed point locus of  $\sigma_i$  on  $\tilde{\mathcal{X}}$  is the disjoint union of smooth curves. They are in one to one correspondence with the irreducible components of the fixed point locus on  $\mathcal{X}$ . The fixed point locus on  $\mathcal{X}$  consists of curves and isolated points which are nodes. If a node occurs as isolated fixed point, then the exceptional line over it is in the fixed point locus on  $\tilde{\mathcal{X}}$ .* FixCur

The image of the fixed point locus of  $\sigma_i$  in  $\tilde{\mathcal{X}}/\sigma_i$  is curve, which is in the singular locus. We claim that in a crepant resolution over each of its components there is only one exceptional divisor. Locally around a fixed point the involution can be described by  $(w_1, w_1, w_3) \mapsto (-w_1, -w_2, w_3)$ . The crepant resolution of the quotient of  $\mathbb{C}^3$  by this involution is easy to describe (we did it in [FSM]) and one sees from this description that the exceptional divisor is smooth and connected.

**7.4 Lemma.** *The Picard number of a crepant resolution of  $\mathcal{X}/\mathcal{G}2\_i$  equals the sum of the Picard number of the regular locus of  $\mathcal{X}/\mathcal{G}2\_i$  (see 7.1) and the number of irreducible components of the fixed point locus of  $\sigma_i$  (considered in  $\mathcal{X}$  is enough).* PicNum

We also have to compute the Euler number of a crepant resolution of  $\mathcal{X}/\mathcal{G}2\_i$ . This is given by the string theoretic Euler number  $e(\tilde{\mathcal{X}}, \mathcal{G}2\_i)$ . We refer to [Re] for some comments about this. We recall the definition of  $e(M, G)$ . Here  $G$  is a finite group acting on a compact differentiable manifold  $M$ . One has to consider the subset of  $G \times G$  of all commuting pairs  $(g, h)$ . Then the string theoretic Euler number is defined as

$$e(M, G) = \frac{1}{\#G} \sum_{gh=hg} e(M^{(g,h)}).$$

Here  $M^{(g,h)}$  denotes the common fixed point set of  $g, h$ . The string theoretic Euler number has the following basic property: Assume that  $M$  is a Calabi Yau manifold and that  $G$  acts by biholomorphic transformations, which leave the Calabi-Yau 3-form invariant. Assume that for  $a \in M$  the stabilizer  $G_a$  acts on the tangent space as subgroup of the special linear group. Then there exists a crepant desingularization of  $M/G$  and for each such desingularization its usual Euler number is  $e(M, G)$ .

We apply this formula in the case, where the order of  $G$  is two. There are 4 commuting pairs  $(e, e)$ ,  $(\sigma, e)$ ,  $(e, \sigma)$ ,  $(\sigma, \sigma)$ .

**7.5 Lemma.** *The Euler number of a crepant resolution of  $\tilde{\mathcal{X}}/\mathcal{G}2.i$  is*

EulBer

$$e = 32 + \frac{3}{2} \sum_C e(C),$$

where  $C$  runs through the components of the fixed point locus of  $\sigma_i$  in  $\tilde{\mathcal{X}}$ .

The fixed point sets are easy to determine. The involution can be considered as a linear transformation  $A : \mathbb{C}^8 \rightarrow \mathbb{C}^8$ , where  $A^2 = aE$ . We want to consider the fixed point locus of  $A$  on  $P^7(\mathbb{C})$ . It corresponds to the eigenspaces of  $A$ . The possible eigenvalues are the two square roots of  $a$ . We denote the two eigenspaces by  $V^+$  and  $V^-$ . Hence  $\mathbb{C}^8 = V^+ \oplus V^-$ . The projective spaces  $P(V^+)$  and  $P(V^-)$  are two disjoint linear subspaces of  $P^7(\mathbb{C}) = P(\mathbb{C}^8)$ . To get the fixed point set of  $A$  inside the variety  $\mathcal{X}$  we have to intersect this variety with the two linear subspaces. Hence the fixed point set inside  $\mathcal{X}$  is the disjoint union of two parts, where each of the parts can be empty of course.

Following these lines one gets:

**7.6 Proposition.** *The following table describes fixed point sets of the involutions  $\sigma_i$ ,  $i \leq i \leq 11$ , on  $\mathcal{X}$  and the Hodge numbers of a Calabi-Yau model of the quotient  $\mathcal{X}/\mathcal{G}2.i$ .*

FixSet

	fixed points	pic	$e$
$\sigma_1$ :	empty set	16	32
$\sigma_2$ :	16 nodes	40	80
$\sigma_3$ :	4 elliptic curves	20	32
$\sigma_4$ :	empty set	16	32
$\sigma_5$ :	empty set	16	32
$\sigma_6$ :	8 conics in planes ( $\cong P^1$ )	28	56
$\sigma_7$ :	8 lines ( $\cong P^1$ )	28	56
$\sigma_8$ :	2 elliptic curves	18	32
$\sigma_9$ :	2 elliptic curves	18	32
$\sigma_{10}$ :	4 conics in planes ( $\cong P^1$ )	22	44

The equations for the fixed point loci can be given explicitly. We just give as an example the 4 elliptic curves which describe the fixed point locus of  $\sigma_3$ : They are described by the following 4 ideals:

$$\begin{aligned}
&(Y_0 + Y_1, \quad Y_2 + Y_3, \quad X_1, \quad X_3, \quad Y_1^2 - X_0^2 - X_2^2, \quad Y_3^2 - X_0^2 + X_2^2), \\
&(Y_0 + Y_1, \quad Y_2 - Y_3, \quad X_1, \quad X_3, \quad Y_1^2 - X_0^2 - X_2^2, \quad Y_3^2 - X_0^2 + X_2^2), \\
&(Y_0 - Y_1, \quad Y_2 + Y_3, \quad X_1, \quad X_3, \quad Y_1^2 - X_0^2 - X_2^2, \quad Y_3^2 - X_0^2 + X_2^2), \\
&(Y_0 - Y_1, \quad Y_2 - Y_3, \quad X_1, \quad X_3, \quad Y_1^2 - X_0^2 - X_2^2, \quad Y_3^2 - X_0^2 + X_2^2).
\end{aligned}$$

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